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# Influence of rapidly oscillating inhomogeneities in the formation of additional boundary layers for singularly perturbed integro-differential systems

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## Abstract

In this paper, the Lomov's regularization method is generalized to a singularly perturbed integro-differential equation with a fractional derivative and with a rapidly oscillating inhomogeneity. The main goal of the work is to reveal the influence of a rapidly changing kernel on the structure of the asymptotic of the problem solution and to study additional boundary functions that are generated by rapidly oscillating inhomogeneities.

*Keywords:* Singularly perturbed Fractional order derivation Integro-differential equation Solvability of iterative problems Rapidly oscillating inhomogeneity.
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## 1. Introduction

The development of an algorithm for the regularization method [1, 2] made it possible to consider problems with a spectrum lying on the imaginary axis [3, 4]. Initially, singularly perturbed differential equations were studied by the regularization method, and then integro-differential equations with slowly varying kernels [5, 6, 7, 8, 9, 10, 11]. The application of this method to singularly perturbed integro-differential equations

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with rapidly changing kernels and to nonlinear equations was not without complications. Since the integral operator generates identically the zero point of the spectrum, along with the resonances induced by the non-zero points of the spectrum, one has to take into account the so-called "zero" resonances, which makes the construction of an asymptotic solution of the problem more complicated. For this reason, linear integro-differential systems with rapidly changing kernels [12, 13, 14, 15] were considered earlier, and nonlinear systems were considered in the case of the absence of resonance [16]. On the basis of these results, the ideas of the regularization method are generalized to singularly perturbed partial differential integro-differential equations with slowly and rapidly varying kernels [16, 18, 19, 20]. However, all the described problems were considered in the absence of fast oscillations, both in coefficients and inhomogeneities.

As is known, when studying the properties of media with a periodic structure, one encounters differential equations with rapidly oscillating coefficients and inhomogeneities. Various methods have been developed for solving such equations, one of which is the splitting method [21, 22, 23] and the regularization method [1, 2]. However, in the splitting method, problems with an integral operator proportional to a small parameter are considered, which significantly narrow the scope of this method. In the well-known works of the regularization method, problems were considered containing only rapidly oscillating coefficients for unknown functions. The generalization of this method to singularly perturbed integro-differential equations and rapidly oscillating components has not been considered before. The influence of such inhomogeneities on the asymptotic of solutions is an interesting and non-trivial problem, which is studied in this paper.

Singularly perturbed differential, integral and integro-differential equations with rapidly oscillating coefficients is considered in [24, 25, 26, 27, 28, 29, 30]. In the papers [31, 32, 33, 34, 35, 36, 37], regularized asymptotic solutions for linear singularly perturbed equations with rapidly changing kernels and rapidly oscillating inhomogeneities are studied and constructed. And in the work [38], an integral equation with a rapidly oscillating inhomogeneity is considered. An integro-differential equation of the Fredholm type with a kernel depending on the solution of a singularly perturbed differential equation was studied in [39]. The presence of the Fredholm integral operator and its type significantly affect the development of an asymptotic solution algorithm, in the implementation of which it is necessary to take into account significant singular singularities generated by the rapidly decreasing kernel of the integral operator. We show that the structure of essentially singular singularities changes when passing from a Volterra-type integral operator to a Fredholm-type operator. If in the case of the Volterra operator they change with a change in the independent variable, then the singularities generated by the kernel of the Fredholm-type integral operator are constant and depend only on a small parameter. Singularly perturbed differential, integro-differential equations with fractional derivatives - in papers [40, 41, 42, 43].

An initial problem is considered for a singularly perturbed integro-differential equation

$$\begin{aligned} L_\varepsilon z(x,\varepsilon) \equiv \varepsilon z^{(\alpha)} - A(x)z - \int\limits_{x_0}^{x} e^{\frac{1}{\varepsilon}\int\limits_s^x \mu(\theta)d\theta} K(x,s)z(s,\varepsilon)ds = \\ = h_1(x) + h_2(x)\sin\tfrac{\beta(x)}{\varepsilon},\ z(x_0,\varepsilon) = z^0,\quad x\in[x_0,X],\quad x_0>0 \end{aligned} \tag{1}$$

for unknown function $z = \{z_1(x,\varepsilon),\ldots,z_n(x,\varepsilon)\}$, $A(x)$ is $(n\times n)-$ matrix, $h_j(x) = (h_{1j}(x),...,h_{nj}(x))$ is the known functions, $\beta'(x)>0,\ \mu(x)<0\ (\forall x\in[x_0,X]),\ 0<\alpha<1,\ z^0-$ constant vector, $\varepsilon>0$ is a small parameter. The problem is posed of constructing a regularized [1, 2] asymptotic solution to problem (1).

According conformable derivative (see, [44]), we rewrite the original fractional order equation (1) in the following form:

$$\begin{aligned} L_\varepsilon z(x,\varepsilon) \equiv \varepsilon x^{(1-\alpha)}\tfrac{dz}{dx} - A(x)z - \int\limits_{x_0}^{x} e^{\frac{1}{\varepsilon}\int\limits_s^x \mu(\theta)d\theta} K(x,s)z(s,\varepsilon)ds = \\ = h_1(x) + h_2(x)\sin\tfrac{\beta(x)}{\varepsilon},\ z(x_0,\varepsilon) = z^0,\ x\in[x_0,X]. \end{aligned} \tag{2}$$

Problem (2) will be considered under the following conditions:

1) $A(x)\in C^\infty\left([x_0,X],\mathbb{C}^{n\times n}\right), h_1(x), h_2(x)\in C^\infty\left([x_0,X],\mathbb{C}^n\right), K(x,s)\in C^\infty\left(\{x_0\le s\le x\le X\},\mathbb{C}^{n\times n}\right);$

2) the spectrum $\{\lambda_1(x), ..., \lambda_n(x)\}$ of the matrix $A(x)$, the frequency $\lambda_{n+1}(x) = -i\beta'(x)$, $\lambda_{n+2}(x) = = +i\beta'(x)$ of the rapidly oscillating inhomogeneity and the spectral value $\lambda_{n+3}(x) = \mu(x)$ of the rapidly varying kernel satisfy at each $t \in [x_0, X]$ the conditions:

*a)* $\lambda_i(x) \neq \lambda_j(x), i \neq j, \lambda_j(x) \neq 0, i, j = \overline{1, n+3}$;

*b)* $\mathrm{Re}\lambda_j(x) < 0, j = \overline{1, n}, \ \mathrm{Re}\lambda_{n+3}(x) \leq 0, \ \beta'(x) > 0.$

## 2. Regularization of the problem (2)

Denote by $\sigma_j = \sigma_j(\varepsilon)$ independent of magnitude $\sigma_1 = e^{-\frac{i}{\varepsilon}\beta(x_0)}$, $\sigma_2 = = e^{+\frac{i}{\varepsilon}\beta(x_0)}$, and introduce the regularized variables:

$$\begin{gathered}\tau_j = \tfrac{1}{\varepsilon}\int\limits_{x_0}^{x} \theta^{(\alpha-1)}\lambda_j(\theta)d\theta \equiv \tfrac{\psi_j(x)}{\varepsilon}, j = \overline{1, n},\\ \tau_i = \tfrac{1}{\varepsilon}\int\limits_{x_0}^{x} \lambda_i(\theta)d\theta \equiv \tfrac{\psi_i(x)}{\varepsilon}, i = \overline{n+1, n+3}\end{gathered} \tag{3}$$

and instead of problem (2), consider the problem

$$\begin{gathered}L_\varepsilon \tilde{z}(x,\tau,\sigma,\varepsilon) \equiv \varepsilon x^{(1-\alpha)}\tfrac{\partial \tilde{z}}{\partial x} + \sum\limits_{j=1}^{n}\lambda_j(x)\tfrac{\partial \tilde{z}}{\partial \tau_j} + x^{(1-\alpha)}\sum\limits_{i=n+1}^{n+3}\lambda_i(x)\tfrac{\partial \tilde{z}}{\partial \tau_i} - A(x)\tilde{z} - \\ -\int\limits_{x_0}^{x} e^{\frac{1}{\varepsilon}\int\limits_{s}^{x}\mu(\theta)d\theta} K(x,s)\tilde{z}(s, \tfrac{\psi(s)}{\varepsilon}, \sigma, \varepsilon)ds = h_1(x) + \tfrac{h_2(x)}{2i}\left(e^{\tau_{n+1}}\sigma_1 - e^{\tau_{n+2}}\sigma_2\right), \ \tilde{z}(x,\tau,\sigma,\varepsilon)|_{x=x_0,\tau=0} = z^0\end{gathered} \tag{4}$$

for the function $\tilde{z} = \tilde{z}(x,\tau,\sigma,\varepsilon)$, where is indicated (according (3)): $\tau = (\tau_1, ..., \tau_{n+3})$, $\psi = (\psi_1, ..., \psi_{n+3})$. It is clear that if $\tilde{z} = \tilde{z}(x,\tau,\sigma,\varepsilon)$ is a solution of the problem (4), then the function is $\tilde{z} = \tilde{z}\left(x, \frac{\psi(x)}{\varepsilon}, \sigma, \varepsilon\right)$ an exact solution to problem (2), therefore, problem (4) is extended with respect to problem (2). However, it cannot be considered fully regularized, since it does not regularize the integral

$$J\tilde{z} \equiv J\left(\tilde{z}(x,\tau,\sigma,\varepsilon)|_{x=s,\tau=\psi(s)/\varepsilon}\right) = \int\limits_{x_0}^{x} e^{\frac{1}{\varepsilon}\int\limits_{s}^{x}\lambda_{n+3}(\theta)d\theta} K(x,s)\tilde{z}(s, \frac{\psi(s)}{\varepsilon}, \sigma, \varepsilon)ds.$$

For its regularization, we introduce the class $M_\varepsilon$ asymptotically invariant with respect to the operator $J\tilde{z}$ (see [1], p. 62]). Consider first the space $U$ of vector functions $z(x,\tau,\sigma)$, representable by the sums

$$z(x,\tau,\sigma) = z_0(x,\sigma) + \sum\limits_{i=1}^{n+3} z_i(x,\sigma)e^{\tau_i}, \ \ z_i\left(x,\sigma\right) \in C^\infty\left(\left[x_0, X\right], \mathbb{C}^n\right), i = \overline{0, n+3}, \tag{5}$$

Note that in (5) the elements of the space $U$ depend on bounded in $\varepsilon > 0$ the constant $\sigma = \sigma(\varepsilon)$, which do not affect the development of the algorithm described below, therefore, henceforth, in the record of element (5) of this space $U$, for the sake of brevity, we omit the dependence on $\sigma$.

Let us show that the class $M_\varepsilon = U|_{\tau=\psi(x)/\varepsilon}$ is asymptotically invariant with respect to the operator $J$. The image of the operator $J$ on the element (5) of the space $U$ has the form:

$$\begin{gathered}Jz\left(x,\tau\right) = \int\limits_{x_0}^{x} e^{\frac{1}{\varepsilon}\int\limits_{s}^{x}\lambda_{n+3}(\theta)d\theta} K\left(x,s\right)z_0\left(s\right)ds + \sum_{j=1}^{n}\int\limits_{x_0}^{x} e^{\frac{1}{\varepsilon}\int\limits_{s}^{x}\lambda_{n+3}(\theta)d\theta} K(x,s)z_j(s)e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{s}\theta^{(\alpha-1)}\lambda_j(\theta)d\theta} ds + \\ + \sum_{i=n+1}^{n+3}\int\limits_{x_0}^{x} e^{\frac{1}{\varepsilon}\int\limits_{s}^{x}\lambda_{n+3}(\theta)d\theta} K\left(x,s\right)z_i\left(s\right)e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{s}\lambda_i(\theta)d\theta} ds = \int\limits_{x_0}^{x} e^{\frac{1}{\varepsilon}\int\limits_{s}^{x}\lambda_{n+3}(\theta)d\theta} K(x,s)z_0(s)ds + \\ + \sum_{j=1}^{n} e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}\lambda_{n+3}(\theta)d\theta} \int\limits_{x_0}^{x} K(x,s)z_j(s)e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{s}[\theta^{(\alpha-1)}\lambda_j(\theta) - \lambda_{n+3}(\theta)]d\theta} ds +\end{gathered}$$

$$+\sum_{i=n+1}^{n+2} e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}\lambda_{n+3}(\theta)d\theta}\int\limits_{x_0}^{x} K\left(x,s\right)z_i\left(s\right)e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{s}(\lambda_i(\theta)-\lambda_{n+3}(\theta))d\theta}ds+e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}\lambda_{n+3}(\theta)d\theta}\int\limits_{x_0}^{x} K(x,s)z_{n+3}(s)ds.$$

Integrating in parts, we have

$$J_0\left(x,\varepsilon\right)=\int\limits_{x_0}^{x} K\left(x,s\right)z_0\left(s\right)e^{\frac{1}{\varepsilon}\int\limits_{s}^{x}\lambda_{n+3}(\theta)d\theta}ds=\varepsilon\int\limits_{x_0}^{x}\frac{K\left(x,s\right)z_0\left(s\right)}{-\lambda_{n+3}\left(s\right)}de^{\frac{1}{\varepsilon}\int\limits_{s}^{x}\lambda_{n+3}(\theta)d\theta}$$

$$=\varepsilon\frac{K\left(x,s\right)z_0\left(s\right)}{-\lambda_{n+3}\left(s\right)}e^{\frac{1}{\varepsilon}\int\limits_{s}^{x}\lambda_{n+3}(\theta)d\theta}\Bigg|_{s=x_0}^{s=x}-\varepsilon\int\limits_{x_0}^{x}\left(\frac{\partial}{\partial s}\frac{K\left(x,s\right)z_0\left(s\right)}{-\lambda_{n+3}\left(s\right)}\right)e^{\frac{1}{\varepsilon}\int\limits_{s}^{x}\lambda_{n+3}(\theta)d\theta}ds=$$

$$=\varepsilon\left[\frac{K\left(x,x_0\right)z_0\left(x_0\right)}{\lambda_{n+3}\left(x_0\right)}e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}\lambda_{n+3}(\theta)d\theta}-\frac{K\left(x,x\right)z_0\left(x\right)}{\lambda_{n+3}\left(x\right)}\right]+\varepsilon\int\limits_{x_0}^{x}\left(\frac{\partial}{\partial s}\frac{K\left(x,s\right)z_0\left(s\right)}{\lambda_{n+3}\left(s\right)}\right)e^{\frac{1}{\varepsilon}\int\limits_{s}^{x}\lambda_{n+3}(\theta)d\theta}ds.$$

Continuing this process further, we obtain the decomposition

$$J_0\left(x,\varepsilon\right)=\sum_{\nu=0}^{\infty}\varepsilon^{\nu+1}\left[\left(I_0^{\nu}\left(K\left(x,s\right)z_0\left(s\right)\right)\right)_{s=x_0}e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}\lambda_{n+3}(\theta)d\theta}-\left(I_0^{\nu}\left(K\left(x,s\right)z_0\left(s\right)\right)\right)_{s=x}\right]$$

where

$$I_0^0=\frac{1}{-\lambda_{n+3}\left(s\right)}\cdot,\ I_0^{\nu}=\frac{1}{-\lambda_{n+3}\left(s\right)}\frac{\partial}{\partial s}I_0^{\nu-1}(\nu\geq 1).$$

Next, apply the same operation to the integrals:

$$J_j(x,\varepsilon)=e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}\lambda_{n+3}(\theta)d\theta}\int\limits_{x_0}^{x} K(x,s)z_j(s)e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{s}[\theta^{(\alpha-1)}\lambda_j(\theta)-\lambda_{n+3}(\theta)]d\theta}ds=$$

$$=\varepsilon e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}\lambda_{n+3}(\theta)d\theta}\int\limits_{x_0}^{x}\frac{K(x,s)z_j(s)}{s^{(\alpha-1)}\lambda_j\left(s\right)-\lambda_{n+3}\left(s\right)}de^{\frac{1}{\varepsilon}\int\limits_{x_0}^{s}[\theta^{(\alpha-1)}\lambda_j(\theta)-\lambda_{n+3}(\theta)]d\theta}ds=$$

$$=\varepsilon e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}\lambda_{n+3}(\theta)d\theta}\left[\frac{K(x,s)z_j(s)}{s^{(\alpha-1)}\lambda_j\left(s\right)-\lambda_{n+3}\left(s\right)}\,e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{s}[\theta^{(\alpha-1)}\lambda_j(\theta)-\lambda_{n+3}(\theta)]d\theta}\Bigg|_{s=x_0}^{s=x}-\right.$$

$$\left.-\int\limits_{x_0}^{x}\left(\frac{\partial}{\partial s}\frac{K(x,s)z_j(s)}{s^{(\alpha-1)}\lambda_j\left(s\right)-\lambda_{n+3}\left(s\right)}\right)e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{s}[\theta^{(\alpha-1)}\lambda_j(\theta)-\lambda_{n+3}(\theta)]d\theta}ds\right]=$$

$$=\varepsilon\left[\frac{K(x,x)z_j(x)}{x^{(\alpha-1)}\lambda_j\left(x\right)-\lambda_{n+3}\left(x\right)}e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}\theta^{(\alpha-1)}\lambda_j(\theta)d\theta}-\frac{K(x,x_0)z_j(x_0)}{x_0^{(\alpha-1)}\lambda_j\left(x_0\right)-\lambda_{n+3}\left(x_0\right)}e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}\lambda_{n+3}(\theta)d\theta}\right]-$$

$$-\varepsilon e^{\frac{1}{\varepsilon}\int\limits_{0}^{x}\lambda_{n+3}(\theta)d\theta}\int\limits_{x_0}^{x}\left(\frac{\partial}{\partial s}\frac{s^{(1-\alpha)}K(x,s)z_j(s)}{\lambda_j\left(s\right)-\lambda_{n+3}\left(s\right)}\right)e^{\frac{1}{\varepsilon}\int\limits_{0}^{s}[\theta^{(\alpha-1)}\lambda_j(\theta)-\lambda_{n+3}(\theta)]d\theta}ds=$$

$$=\sum_{\nu=0}^{\infty}(-1)^{\nu}\varepsilon^{\nu+1}\left[\left(I_j^{\nu}\left(K\left(x,s\right)z_j\left(s\right)\right)\right)_{s=x}e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}\theta^{(\alpha-1)}\lambda_j(\theta))d\theta}-\right.$$

$$-\left(I_j^{\nu}\left(K\left(x,s\right)z_j\left(s\right)\right)\right)_{s=x_0} e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}\lambda_{n+3}(\theta)d\theta}\Bigg],\ j=\overline{1,n}$$

where

$$I_j^0=\frac{1}{s^{(\alpha-1)}\lambda_j\left(s\right)-\lambda_{n+3}\left(s\right)}\cdot,\ I_j^{\nu}=\frac{1}{s_0^{(\alpha-1)}\lambda_j\left(s\right)-\lambda_{n+3}\left(s\right)}\frac{\partial}{\partial s}I_j^{\nu-1}(\nu\geq 1).$$

Next, apply the same operation to the integrals:

$$J_i\left(x,\varepsilon\right)=e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}\lambda_{n+3}(\theta)d\theta}\int\limits_{x_0}^{x}K\left(x,s\right)z_i\left(s\right)e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{s}(\lambda_i(\theta)-\lambda_{n+3}(\theta))d\theta}ds=$$

$$=\varepsilon e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}\lambda_{n+3}(\theta)d\theta}\int\limits_{x_0}^{x}\frac{K(x,s)z_i(s)}{\lambda_i(s)-\lambda_{n+3}(s)}de^{\frac{1}{\varepsilon}\int\limits_{x_0}^{s}(\lambda_i(\theta)-\lambda_{n+3}(\theta))d\theta}=$$

$$=\varepsilon e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}\lambda_{n+3}(\theta)d\theta}\left[\frac{K(x,s)z_i(s)}{\lambda_i(s)-\lambda_{n+3}(s)}e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{s}(\lambda_i(\theta)-\lambda_{n+3}(\theta))d\theta}\Bigg|_{s=x_0}^{s=x}-\right.$$

$$\left.-\varepsilon\int\limits_{x_0}^{x}\left(\frac{\partial}{\partial s}\frac{K\left(x,s\right)z_i\left(s\right)}{\lambda_i\left(s\right)-\lambda_{n+3}\left(s\right)}\right)e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{s}(\lambda_i(\theta)-\lambda_{n+3}(\theta))d\theta}ds\right]=$$

$$=\varepsilon\left[\frac{K\left(x,x\right)z_i\left(x\right)}{\lambda_i\left(x\right)-\lambda_{n+3}\left(x\right)}e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}\lambda_i(\theta)d\theta}-\frac{K\left(x,x_0\right)z_i\left(x_0\right)}{\lambda_i\left(x_0\right)-\lambda_{n+3}\left(x_0\right)}e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}\lambda_{n+3}(\theta)d\theta}\right]-$$

$$-\varepsilon e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}\lambda_{n+3}(\theta)d\theta}\int\limits_{x_0}^{x}\left(\frac{\partial}{\partial s}\frac{K\left(x,s\right)z_i\left(s\right)}{\lambda_i\left(s\right)-\lambda_{n+3}\left(s\right)}\right)e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{s}(\lambda_i(\theta)-\lambda_{n+3}(\theta))d\theta}ds=$$

$$=\sum_{\nu=0}^{\infty}(-1)^{\nu}\varepsilon^{\nu+1}\left[\left(I_i^{\nu}\left(K\left(x,s\right)z_i\left(s\right)\right)\right)_{s=x}e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}\lambda_i(\theta))d\theta}-\left(I_i^{\nu}\left(K\left(x,s\right)z_i\left(s\right)\right)\right)_{s=x_0}e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}\lambda_{n+3}(\theta)d\theta}\right]$$

where

$$I_i^0=\frac{1}{\lambda_i\left(s\right)-\lambda_{n+3}\left(s\right)}\cdot,\ I_i^{\nu}=\frac{1}{\lambda_i\left(s\right)-\lambda_{n+3}\left(s\right)}\frac{\partial}{\partial s}I_i^{\nu-1}\left(\nu\geq 1\right),i=n+1,n+2.$$

This means that the image of the operator $J$ on the element (5) of the space $U$ is represented as a series

$$Jz\left(x,\tau\right)=e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}\lambda_{n+3}(\theta))d\theta}\int\limits_{x_0}^{x}K\left(x,s\right)z_{n+3}\left(s\right)ds+$$

$$+\sum_{\nu=0}^{\infty}\varepsilon^{\nu+1}\left[\left(I_0^{\nu}\left(K\left(x,s\right)z_0\left(s\right)\right)\right)_{s=x_0}e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}\lambda_{n+3}(\theta))d\theta}-\left(I_0^{\nu}\left(K\left(x,s\right)z_0\left(s\right)\right)\right)_{s=x}\right]+$$

$$+\sum_{j=1}^{n}\sum_{\nu=0}^{\infty}\varepsilon^{\nu+1}\left[\left(I_j^{\nu}\left(K\left(x,s\right)z_j\left(s\right)\right)\right)_{s=x}e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}\theta^{(\alpha-1)}\lambda_j(\theta)}d\theta-\left(I_j^{\nu}\left(K\left(x,s\right)z_j\left(s\right)\right)\right)_{s=x_0}e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}\lambda_{n+3}(\theta)d\theta}\right]+$$

$$+\sum_{i=2}^{3}\sum_{\nu=0}^{\infty}\varepsilon^{\nu+1}\left[\left(I_i^{\nu}\left(K\left(x,s\right)z_i\left(s\right)\right)\right)_{s=x}e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}\lambda_i(\theta))d\theta}-\left(I_i^{\nu}\left(K\left(x,s\right)z_i\left(s\right)\right)\right)_{s=x_0}e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}\lambda_{n+3}(\theta)d\theta}\right].$$

It is easy to show (see, for example, [45], pp. 291-294) that this series converges asymptotically for $\varepsilon\to+0$ (uniformly in $x\in[x_0,X]$). This means that the class $M_\varepsilon$ is asymptotically invariant (for $\varepsilon\to+0$) with respect to the operator $J$.

We introduce operators $R_\nu: U\to U$, acting on each element $z(x,\tau)\in U$ of the form (5) according to the law:

$$R_0z(x,\tau)=e^{\tau_{n+3}}\int\limits_{x_0}^{x}K(x,s)z_{n+3}(s)ds, \tag{$6_0$}$$

$$\begin{gathered}R_1z(x,\tau)=\left[\left(I_0^0\left(K\left(x,s\right)z_0\left(s\right)\right)\right)_{s=x_0}e^{\tau_{n+3}}-\left(I_0^0\left(K\left(x,s\right)z_0\left(s\right)\right)\right)_{s=x}\right]+\\+\sum_{j=1}^{n}\left[\left(I_j^0\left(K\left(x,s\right)z_j\left(s\right)\right)\right)_{s=x_0}e^{\tau_j}-\left(I_j^0\left(K\left(x,s\right)z_j\left(s\right)\right)\right)_{s=x_0}e^{\tau_{n+3}}\right]+\\+\sum_{i=1}^{n+2}\left[\left(I_i^0\left(K(x,s)z_i(s)\right)\right)_{s=x}e^{\tau_i}-\left(I_i^0\left(K(x,s)z_i(s)\right)\right)_{s=x_0}e^{\tau_{n+3}}\right],\end{gathered}\tag{$6_1$}$$

$$\begin{gathered}R_{\nu+1}z\left(x,\tau\right)=(-1)^{\nu}\left[\left(I_0^{\nu}\left(K\left(x,s\right)z_0\left(s\right)\right)\right)_{s=x_0}e^{\tau_{n+3}}-\left(I_0^{\nu}\left(K\left(x,s\right)z_0\left(s\right)\right)\right)_{s=x}\right]+\\+\sum_{j=1}^{n}\left[\left(I_j^{\nu}\left(K\left(x,s\right)z_j\left(s\right)\right)\right)_{s=x_0}e^{\tau_j}-\left(I_j^{\nu}\left(K\left(x,s\right)z_j\left(s\right)\right)\right)_{s=x_0}e^{\tau_{n+3}}\right]+\\+\sum_{i=1}^{n+2}(-1)^{\nu}\left[\left(I_i^{\nu}\left(K(x,s)z_i(s)\right)\right)_{s=x}e^{\tau_i}-\left(I_i^{\nu}\left(K(x,s)z_i(s)\right)\right)_{s=x_0}e^{\tau_{n+3}}\right].\end{gathered}\tag{$6_{\nu+1}$}$$

Now let $\tilde{z}(x,\tau,\varepsilon)$ be an arbitrary continuous function on $(x,\tau)\in G==[x_0,X]\times\{\tau:Re\tau_j\leq0,\ j==\overline{1,n+3}\}$, with asymptotic expansion

$$\tilde{z}(x,\tau,\varepsilon)=\sum_{k=0}^{\infty}\varepsilon^k z_k(x,\tau),\ \ z_k(x,\tau)\in U \tag{7}$$

converging as $\varepsilon\to+0$ (uniformly in $(x,\tau)\in G$). Then the image $J\tilde{z}\left(x,\tau,\varepsilon\right)$ of this function is decomposed into an asymptotic series

$$J\tilde{z}(x,\tau,\varepsilon)=\sum_{k=0}^{\infty}\varepsilon^kJz_k(x,\tau)=\sum_{r=0}^{\infty}\varepsilon^r\sum_{s=0}^{r}R_{r-s}z_s(x,\tau)|_{\tau=\psi(x)/\varepsilon}.$$

This equality is the basis for introducing an extension of an operator $J$ on series of the form (7):

$$\tilde{J}\tilde{z}\equiv\tilde{J}\left(\sum_{k=0}^{\infty}\varepsilon^kz_k(x,\tau)\right)=\sum_{r=0}^{\infty}\varepsilon^r\left(\sum_{k=0}^{r}R_{r-k}z_k(x,\tau)\right).$$

Although the operator $\tilde{J}$ is formally defined, its utility is obvious, since in practice it is usual to construct the $N$-th approximation of the asymptotic solution of the problem (2), in which impose only $N$-th partial sums of the series (7), which have not a formal, but a true meaning. Now you can write a problem that is completely regularized with respect to the original problem (2):

$$\begin{gathered}L_\varepsilon\tilde{z}(x,\tau,\sigma,\varepsilon)\equiv\varepsilon x^{(1-\alpha)}\frac{\partial\tilde{z}}{\partial t}+\sum\limits_{j=1}^{n}\lambda_j(x)\frac{\partial\tilde{z}}{\partial\tau_j}+x^{(1-\alpha)}\sum\limits_{i=n+1}^{n+3}\lambda_i(x)\frac{\partial\tilde{z}}{\partial\tau_i}-\\-A(x)\tilde{z}-\tilde{J}\tilde{z}=h_1(x)-\frac{1}{2i}h_2(x)\left(e^{\tau_{n+1}}\sigma_1-e^{\tau_{n+2}}\sigma_2\right),\ \ \tilde{z}(x_0,0,\sigma,\varepsilon)=z^0\ \ x\in[x_0,X].\end{gathered}\tag{8}$$

## 3. Solvability of the general iterative problem in the space $U$

Substituting the series (7) into (8) and equating the coefficients of the same powers of $\varepsilon$, we obtain the following iterative problems:

$$\begin{gathered}Lz_0(x,\tau,\sigma)\equiv\sum_{j=1}^{n}\tfrac{\partial z_0}{\partial\tau_j}+x^{(1-\alpha)}\sum_{i=n+1}^{n+3}\lambda_i(x)\tfrac{\partial z_0}{\partial\tau_i}-A(x)z_0-R_0z_0=\\=h_1(x)-\tfrac{1}{2i}h_2(x)\left(e^{\tau_{n+1}}\sigma_1-e^{\tau_{n+2}}\sigma_2\right),\ \ z_0(x_0,0)=z^0;\end{gathered}\tag{$9_0$}$$

$$Lz_1(x,\tau,\sigma)=-x^{(1-\alpha)}\frac{\partial z_0}{\partial x}+R_1z_0,\quad z_1(x_0,0)=0;\tag{$9_1$}$$

$$Lz_2(x,\tau,\sigma)=-x^{(1-\alpha)}\frac{\partial z_1}{\partial x}+R_1z_1+R_2z_0,\quad z_2(x_0,0)=0;\tag{$9_2$}$$

..........................................................

$$L\,z_k(x,\tau,\sigma)=-x^{(1-\alpha)}\frac{\partial z_{k-1}}{\partial x}+R_kz_0+\ldots+R_1z_{k-1},z_k(x_0,0)=0,k\geq 1.\tag{$9_k$}$$

Each iterative problem $(9_k)$ has the form

$$L\,z(x,\tau,\sigma)\equiv\sum_{j=1}^{n}\tfrac{\partial z}{\partial\tau_j}+x^{(1-\alpha)}\sum_{i=n+1}^{n+3}\lambda_i(x)\tfrac{\partial z}{\partial\tau_i}-A(x)z-R_0z=H(x,\tau,\sigma),\ \ z(x_0,0)=z_*\tag{10}$$

where $H(x,\tau,\sigma)=H_0(x,\sigma)+\sum\limits_{i=1}^{n+3}H_i(x,\sigma)e^{\tau_i}$ is the known function of space $U$, $z_*-$ is a the known constant vector of the complex space $\mathbb{C}^n$, and the operator $R_0$ has the form (see $(6_0)$)

$$R_0z\equiv R_0\left(z_0(x,\sigma)+\sum_{i=1}^{n+3}z_i(x,\sigma)e^{\tau_i}\right)\triangleq e^{\tau_{n+3}}\int_{x_0}^{x}K(x,s)z_{n+3}(s)ds.$$

We introduce scalar (for each $x\in[x_0,X]$) product in space $U$:

$$<u,w>\equiv<u_0(x)+\sum_{j=1}^{n+3}u_j(x)e^{\tau_j},w_0(x)+\sum_{j=1}^{n+3}w_j(x)e^{\tau_j}>\equiv\sum_{j=0}^{n+3}(u_j(x),w_j(x))$$

where we denote by $(*,*)$ the usual scalar product in the complex space $\mathbb{C}^n:\ (u,v)=u\cdot\bar{v}$. Let us prove the following statement.

**Theorem 3.1.** *Let conditions 1), 2a) be fulfilled and the right-hand side* $H(x,\tau,\sigma)=H_0(x,\sigma)+\sum\limits_{j=1}^{n+3}H_j(x,\sigma)e^{\tau_j}$ *of equation (10) belongs to the space* $U$*. Then the equation (10) is solvable in* $U$*, if and only if*

$$<H(x,\tau),\chi_i(x)e^{\tau_i}>\equiv 0,\ \ i=\overline{1,n}\ \ \forall x\in[x_0,X].\tag{11}$$

*where* $\chi_i(x)$ *are the eigenvectors of the adjoint matrix* $A^*(x)$*, corresponding to the eigenvalues* $\overline{\lambda_i}(x),i=\overline{1,n}$*.*

*Proof.* We will determine the solution of equation (10) as an element (5) of the space $U$ :

$$z(x,\tau,\sigma)=z_0(x,\sigma)+\sum_{j=1}^{n+3}z_j(x,\sigma)e^{\tau_j}.\tag{12}$$

Substituting (12) into equation (10), and equating here the free terms and coefficients separately for identical exponents, we obtain the following equations of equations:

$$-A(x)z_0(x,\sigma)=H_0(x,\sigma),\tag{$13_0$}$$

$$[\lambda_j(x)I - A(x)]\, z_j(x) = H_j(x,\sigma),\ \ j = \overline{1,n}, \tag{$13_j$}$$

$$\left[x^{(1-\alpha)}\lambda_i(x)I - A(x)\right] z_i(x,\sigma) = H_i(x,\sigma),\ \ i = n+1, n+2, \tag{$13_i$}$$

$$\left[x^{(1-\alpha)}\lambda_{n+3}(x)I - A(x)\right] z_{n+3}(x,\sigma) - \int\limits_{x_0}^{x} K(x,s) z_{n+3}(s,\sigma)ds = H_{n+3}(x,\sigma). \tag{$13_{n+3}$}$$

Since the $det A(x) \neq 0$, the system ($13_0$) has a solution

$$z_0(x,\sigma) = -A^{-1}(x)H_0(x,\sigma) \in C^{\infty}\left([x_0, X], \mathbb{C}^n\right).$$

Since $\det[x^{(1-\alpha)}\lambda_i(x)I - A(x)] \neq 0\ \ \forall x \in [x_0, X], i = \overline{n+1, n+2}$, the systems ($13_i$) has a solution

$$z_i(x,\sigma = \left[x^{(1-\alpha)}\lambda_i(x)I - A(x)\right]^{-1} H_i(x,\sigma) \in C^{\infty}\left([x_0, X], \mathbb{C}^n\right),\ \ i = n+1, n+2.$$

Since $\det[x^{(1-\alpha)}\lambda_{n+3}(x)I - A(x)] \neq 0\ \ \forall x \in [x_0, X]$, the system ($13_{n+3}$) can be written in the form

$$\begin{aligned} z_{n+3}\,(x) = \int\limits_{x_0}^{x} \left(\left[x^{(1-\alpha)}\lambda_{n+3}(x)I - A(x)\right]^{-1} K(x,s)\right) z_{n+3}(s)ds + \\ + \left[x^{(1-\alpha)}\lambda_{n+3}(x)I - A(x)\right]^{-1} H_{n+3}(x). \end{aligned} \tag{14}$$

Due to the smoothness of the kernel $\left[x^{(1-\alpha)}\lambda_{n+3}(x)I - A(x)\right]^{-1} K(x,s)$ and inhomogeneity $\left[x^{(1-\alpha)}\lambda_{n+3}(x)I - -A(x)\right]^{-1} H_{n+3}(x)$, this Volterra integral system has unique solution $z_{n+3}(x) \in C^{\infty}\left([x_0, X], \mathbb{C}^n\right)$.

The system ($13_j$) at $j = 1, 2, ..., n$ is solvable in the space $C^{\infty}\left([x_0, X], \mathbb{C}^n\right)$ if and only if the identities

$$< H(x,\tau), \chi_j(x) > \equiv 0, \quad j = \overline{1,n}\ \ \forall x \in [x_0, X]$$

hold. It is easy to see that these identities coincide with the identities (11). Thus, condition (11) is necessary and sufficient for the solvability of system (10) in the space $U$. □

Let $\varphi_k(x)$ be the eigenvectors of the matrix $A(x)$, corresponding to the eigenvalues $\lambda_k(x), k = \overline{1,n}$. Moreover, the system $\{\varphi_k(x)\}$ is biortonormal with respect to the system $\{\chi_j(x)\}$, i.e.

$$(\varphi_k(x), \chi_j(x)) = \left\{ \begin{array}{l} 1, k = j, \\ 0, k \neq j. \end{array} \right.$$

**Remark 3.2.** *If identity (11) holds, then under conditions 1), 2), equation (10) has the following solution in the space* $U$ :

$$\begin{aligned} z(x,\tau,\sigma) = z_0(x,\sigma) + \textstyle\sum_{k=1}^{n} \left[\eta_k(x)\varphi_k(x) + \sum_{s=1, s\neq k}^{n} \frac{(H_k(x), \chi_s(x))}{x^{(\alpha-1)}\lambda_k(x) - \lambda_s(x)}\varphi_s(x)\right] e^{\tau_k} + \\ + \sum_{j=n+1}^{n+2} \left[x^{(1-\alpha)}\lambda_j(x) - \lambda_{n+3}(x)\right]^{-1} H_j(x,\sigma)e^{\tau_j} + z_{n+3}(x,\sigma)e^{\tau_{n+3}} \end{aligned} \tag{15}$$

*where* $\eta_k(x,\sigma) \in C^{\infty}\left([x_0, X], \mathbb{C}^n\right)$, *are arbitrary function,* $k = \overline{1,n}$, $\ z_{n+3}(x,\sigma)$ *is the solution of an integral equation* (14).

## 4. The unique solvability of the general iterative problem in the space $U$. Remaining term theorem

As can be seen from (15), the solution to system (10) is determined ambiguously. However, if it is subject to additional conditions:

$$\begin{aligned} & z(x_0, 0) = z^*, \\ & < -x^{(1-\alpha)}\tfrac{\partial z}{\partial x} + R_1 y + Q(x,\tau), \chi_k(x)e^{\tau_k} > \equiv 0,\ k = \overline{1,n}, \end{aligned} \tag{16}$$

where $Q(x,\tau)=Q_0(x)+\sum\limits_{j=1}^{n+3}Q_j(x)e^{\tau_j}$ is the known function of the space $U$, $z^*-$ is a constant vector of the complex space $\mathbb{C}^n$, then problem (10) will be uniquely solvable in the space $U$. More precisely, the following result holds.

**Theorem 4.1.** *Let conditions 1) and 2a) be satisfied, the right-hand side $H(x,\tau)$ of the system (10) belongs to the space $U$ and satisfies the orthogonality conditions (11). Then system (10) under additional conditions (16) is uniquely solvable in $U$.*

*Proof.* Under condition (11), system (10) has a solution (15) in the space $U$, where the functions $\eta_k(x)\in C^\infty([x_0,X],\mathbb{C}^n)\,,k=\overline{1,n}$ are still arbitrary. Subordinate (15) to the first condition (16), i.e. $z(x_0,0)=z_*$. We obtain $\sum\limits_{k=1}^{n}\eta_k(x_0)\varphi_k(x_0)=z^*$, where is denoted:

$$z^*=z_*+A^{-1}(x_0)H_0(x_0)-\sum\limits_{k=1}^{n}\sum\limits_{s=1,s\neq k}^{n}\tfrac{(H_k(x_0),\chi_s(x_0))}{x_0^{(\alpha-1)}\lambda_k(x_0)-\lambda_s(x_0)}\varphi_s(x_0)-$$

$$-\left[\lambda_{n+1}(x_0)I-A(x_0)\right]^{-1}H_{n+1}(x_0)-\left[\lambda_{n+2}(x_0)I-A(x_0)\right]^{-1}H_{n+2}(x_0).$$

Multiplying the scalar equality $\sum\limits_{k=1}^{n}\eta_k(x_0)\varphi_k(x_0)=z^*$ by $\chi_s(x_0)$ and taking into account the biorthogonality of the systems $\{\varphi_k(x)\}$ and $\{\chi_j(x)\}$, we find the values

$$\eta_k(x_0)=\left(z^*,\chi_s(x_0)\right),k=\overline{1,n}. \tag{17}$$

Let us now subject solution (15) to the second condition (16). The right side of this equation:

$$-x^{(1-\alpha)}\frac{\partial x_0}{\partial x}+R_1z_0+Q\left(x,\tau\right)=-z_0'(x)-\sum_{k=1}^{n}\left(\alpha_k(x)\varphi_k(x)\right)'e^{\tau_k}-$$

$$-\sum_{s=1,s\neq k}^{n}\left(\frac{(H_k(x),\chi_s(x))}{x^{(\alpha-1)}\lambda_k(x)-\lambda_s(x)}\varphi_s(t)\right)'e^{\tau_k}-\left([\lambda_{n+1}(x)I-A(x)]^{-1}H_{n+1}(x)\right)'e^{\tau_{n+1}}-$$

$$-\left([\lambda_{n+2}(x)I-A(x)]^{-1}H_{n+2}(x)\right)'e^{\tau_{n+2}}+z_{n+3}'(x)e^{\tau_{n+3}}+$$

$$+\sum_{k=1}^{n}\left[\frac{K(x,x)\eta_k(x)\varphi_k(x)}{x^{(\alpha-1)}\lambda_k(x)-\lambda_{n+3}(x)}\,e^{\tau_k}-\frac{K(x,x_0)\eta_k(x_0)\varphi_k(x_0)}{x_0{}^{(\alpha-1)}\lambda_k(x_0)-\lambda_{n+3}(x_0)}e^{\tau_{n+3}}\right]+$$

$$+\frac{K(x,x)z_{n+1}(x)}{\lambda_{n+1}(x)-\lambda_{n+3}(x)}e^{\tau_{n+1}}-\frac{K(x,x_0)z_{n+1}(x_0)}{\lambda_{n+1}(x_0)-\lambda_{n+3}(x_0)}e^{\tau_{n+3}}+\frac{K(x,x)z_{n+2}(x)}{\lambda_{n+2}(x)-\lambda_{n+3}(x)}e^{\tau_{n+2}}- \tag{18}$$

$$-\frac{K(x,x_0)z_{n+2}(x_0)}{\lambda_{n+2}(x_0)-\lambda_{n+3}(x_0)}e^{\tau_{n+3}}+\frac{K(x,x_0)z_0(x_0)}{\lambda_{n+3}(x_0)}e^{\tau_{n+3}}-\frac{K(x,x)z_0(x)}{\lambda_{n+3}(x)}+Q(x,\tau).$$

Now multiplying (18) scalarly by $\chi_k(x)e^{\tau_k},\ k=\overline{1,n}$, we obtain the system of ordinary differential equations

$$\eta_k'(x)+[(\varphi_k'(x),\chi_k(x))+\sum\limits_{s=1,s\neq k}^{n}\left(\tfrac{(H_k(x),\chi_s(t))}{x^{(\alpha-1)}\lambda_k(x)-\lambda_s(x)}\right)(\varphi_s'(x),\chi_k(x))-$$

$$-\tfrac{K(x,x)}{x^{(\alpha-1)}\lambda_k(x)-\lambda_{n+3}(x)}(\varphi_k(x),\chi_k(x))]\eta_k(x)+(Q_k(x),\chi_k(x))=0,k=\overline{1,n}.$$

Adding the initial condition (17) to it, we can uniquely find the functions $\alpha_k(x),k=\overline{1,n}$ and, hence, construct a solution (15) of the problem (10) in the space $U$ in a unique way. ☐

Applying Theorems 3.1 and 4.1 to iterative problems $(9_k)$, we find uniquely their solutions in the space $U$ and construct series (7). Just as in [45], we prove the following statement.

**Theorem 4.2.** *Let conditions 1) – 2) be satisfied for system (2). Then, for* $\varepsilon \in (0, \varepsilon_0]$*(*$\varepsilon_0 > 0$ *is sufficiently small), system (2) has a unique solution* $z(x, \varepsilon) \in \mathrm{C}^1([x_0, X], \mathbb{C}^n)$*; in this case, the estimate*

$$||z(x,\varepsilon) - z_{\varepsilon N}(x)||_{C[x_0,X]} \le c_N \varepsilon^{N+1}, N = 0, 1, 2, \dots$$

*holds. Here* $z_{\varepsilon N}(x)$ *is the restriction (at* $\tau = \frac{\psi(x)}{\varepsilon}$*) of the N-th partial sum of the series (7) (with coefficients* $z_k(x, \tau) \in U$*, satisfying the iterative problems* $(9_k)$*), and the constant* $c_N > 0$ *does not depend on* $\varepsilon$ *at* $\varepsilon \in (0, \varepsilon_0]$*.*

## 5. Construction of the solution of the first iteration problem

Using Theorem 1, we will try to find a solution to the first iteration problem $(9_0)$. Since the right-hand side $h_1(x) - \frac{1}{2i} h_2(x) \left(e^{\tau_{n+1}}\sigma_1 - e^{\tau_{n+2}}\sigma_2\right)$ of the system $(9_0)$, satisfy condition (11), this system has (according to (15)) a solution in the space $U$ in the form

$$z_0(x, \tau) = z_0^{(0)}(x) + \sum_{k=1}^{n} \eta_k(x)\varphi_k(x)e^{\tau_k} + z_{n+1}(x)\sigma_1 e^{\tau_{n+1}} + z_{n+2}(x)\sigma_2 e^{\tau_{n+2}} \tag{19}$$

where $\eta_k^{(0)}(x) \in C^\infty\left([x_0, X], \mathbb{C}^n\right)$ are arbitrary functions, $k = \overline{1, n}$, $\;z_0^{(0)}(x) = -A^{-1}(x)h_1(x)$,

$$z_{n+1}(x) = -\frac{1}{2i}\frac{h_2(x)}{x^{(1-\alpha)}\lambda_{n+1}(x) - \lambda_{n+3}(x)}, \quad z_{n+2}(x) = \frac{1}{2i}\frac{h_2(x)}{x^{(1-\alpha)}\lambda_{n+2}(x) - \lambda_{n+3}(x)}.$$

Submitting (19) to the initial condition $z_0(x_0, 0) = z^0$, we will have

$$z_0^{(0)}(x_0) + \sum_{k=1}^{n} \eta_k^{(0)}(x_0)\varphi_k(x_0) + z_{n+1}^{(0)}(x_0)\sigma_1 + z_{n+2}^{(0)}(x_0)\sigma_2 = z^0 \Leftrightarrow$$

$$\Leftrightarrow \sum_{k=1}^{n} \eta_k^{(0)}(x_0)\varphi_k(x_0) = z^0 + A^{-1}(x_0)h_1(x_0) - z_{n+1}^{(0)}(x_0)\sigma_1 - z_{n+2}^{(0)}(x_0)\sigma_2.$$

Multiplying this equality scalarly by $\chi_j(x_0), j = \overline{1, n}$, uniquely we find $\alpha_k^{(0)}(x_0)$ :

$$\eta_k^{(0)}(x_0) = \left(\left(z^0 + A^{-1}(x_0)h_1(x_0) - z_{n+1}(x_0)\sigma_1 - z_{n+2}(x_0)\sigma_2\right), \chi_j(x_0)\right), k = \overline{1, n}. \tag{20}$$

For a complete calculation of the functions $\eta_k^{(0)}(t)$, we pass to the next iterative problem $(9_1)$. Substituting the solution (19) of the system $(9_0)$ into it, we obtain the following system of equations:

$$Lz_1(x, \tau) = -x^{(1-\alpha)}\frac{d}{dx}z_0^{(0)}(x) - x^{(1-\alpha)}\sum_{k=1}^{n}\frac{d}{dx}\left(\eta_k^{(0)}(x)\varphi_k(x)\right)e^{\tau_k} -$$

$$-x^{(1-\alpha)}\frac{d}{dx}(z_{n+1}^{(0)}(x))\sigma_1 e^{\tau_{n+1}} - x^{(1-\alpha)}\frac{d}{dx}(z_{n+2}^{(0)}(x))\sigma_2 e^{\tau_{n+2}} + R_1 z_0 =$$

$$= -x^{(1-\alpha)}\dot{z}_0^{(0)}(x) - x^{(1-\alpha)}\sum_{k=1}^{n}\left(\dot{\eta}_k^{(0)}(x)\varphi_k(x) + \eta_k^{(0)}(x)\dot{\varphi}_k(x)\right)e^{\tau_k} -$$

$$-x^{(1-\alpha)}\dot{z}_{n+1}^{(0)}(x)\sigma_1 e^{\tau_{n+1}} - x^{(1-\alpha)}\dot{z}_{n+2}^{(0)}(x)\sigma_2 e^{\tau_{n+2}} +$$

$$+\sum_{k=1}^{n}\left[\frac{K(x,x)\eta_k^{(0)}(x)}{x^{(\alpha-1)}\lambda_k(x)-\lambda_{n+3}(x)}\varphi_k(x)e^{\tau_k}-\frac{K(x,x_0)\eta_k^{(0)}(x_0)}{x_0{}^{(\alpha-1)}\lambda_k(x_0)-\lambda_{n+3}(x_0)}\varphi_k(x_0)e^{\tau_{n+3}}\right]+$$

$$+\frac{K(x,x)z_{n+1}^{(0)}(x)}{\lambda_{n+1}(x)-\lambda_{n+3}(x)}e^{\tau_{n+1}}-\frac{K(x,0)z_{n+1}^{(0)}(x_0)}{\lambda_{n+1}(x_0)-\lambda_{n+3}(x_0)}e^{\tau_{n+3}}+$$

$$+\frac{K(x,x)z_{n+2}^{(0)}(x)}{\lambda_{n+2}(x)-\lambda_{n+3}(x)}e^{\tau_{n+2}}-\frac{K(x,0)z_{n+2}^{(0)}(x_0)}{\lambda_{n+2}(x_0)-\lambda_{n+3}(x_0)}e^{\tau_{n+3}}.$$

Performing here scalar multiplication by $\chi_j(x), j=\overline{1,n}$, we obtain the following system of ordinary differential equations

$$x^{(1-\alpha)}\frac{d\eta_k^{(0)}(x)}{x}+\left[(\dot{\varphi}_k(x),\chi_j(x))-\frac{K(x,x)}{x^{(\alpha-1)}\lambda_k(x)-\lambda_{n+3}(x)}(\varphi_k(x),\chi_j(x))\right]\eta_k^{(0)}(x)=0, k=\overline{1,n}.$$

Adding the initial condition (20) to this equation, we find $\eta_k^{(0)}(x):$

$$\eta_k^{(0)}(x)=\eta_k^{(0)}(x_0)e^{-\int\limits_{x_0}^{x}\frac{\left[\left({\varphi'}_k(\xi),\chi_j(\xi)\right)-\frac{K(\xi,\xi)}{\xi^{(\alpha-1)}\lambda_k(\xi)-\lambda_{n+2}(\xi)}\left(\varphi_k(\xi),\chi_j(\xi)\right)\right]}{\xi^{(1-\alpha)}}d\xi}, k,j=\overline{1,n}$$

and hence the solution (19) of the problem $(9_0)$ will be found uniquely in the space $U$. In this case, the leading term of the asymptotics has the following form:

$$\begin{gathered}
z_{\varepsilon 0}(x)=-A^{-1}(x)h_1(x)+\sum_{k=1}^{n}\left[\left(z^0+A^{-1}(x_0)h_1(x_0)+\right.\right.\\
\left.\left.+\frac{1}{2i}\frac{h_2(x_0)}{x_0^{(1-\alpha)}}\left(\frac{\sigma_1}{\lambda_{n+1}(x_0)-\lambda_{n+3}(x_0)}-\frac{\sigma_2}{\lambda_{n+2}(x_0)-\lambda_{n+3}(x_0)}\right)\right),\chi_j(x_0)\right]\times\\
\times\varphi_k(x)e^{-\int\limits_{x_0}^{x}\frac{\left[\left({\varphi'}_k(\xi),\chi_j(\xi)\right)-\frac{K(\xi,\xi)}{\xi^{(\alpha-1)}\lambda_k(\xi)-\lambda_{n+2}(\xi)}\left(\varphi_k(\xi),\chi_j(\xi)\right)\right]}{\xi^{(1-\alpha)}}d\xi+\frac{1}{\varepsilon}\int\limits_0^x\theta^{(\alpha-1)}\lambda_j(\theta)d\theta}-\\
-\frac{1}{2i}\frac{h_2(x)}{\lambda_{n+1}(x)-\lambda_{n+3}\left(x\right)}\sigma_1 e^{\frac{1}{\varepsilon}\int\limits_0^x x^{(1-\alpha)}\lambda_{n+1}(\theta)d\theta}+\\
+\frac{1}{2i}\frac{h_2(x)}{x^{(1-\alpha)}\lambda_{n+2}(x)-\lambda_{n+3}\left(x\right)}\sigma_2 e^{\frac{1}{\varepsilon}\int\limits_0^x\lambda_{n+2}(\theta)d\theta}, k,j=\overline{1,n}.
\end{gathered}\tag{21}$$

## 6. Conclusion

It can be seen from expression (21) for $z_{\varepsilon 0}(x)$ that the main term of the asymptotics of problem (2) depends on rapidly oscillating inhomogeneities (see the exponent at $e^{\tau_{n+1}}, e^{\tau_{n+2}}$) and the fractional derivative participates at the boundary layer (see, the exponent at $e^{\tau_j}, j=\overline{1,n}$).

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